7.SP Stay or Switch?

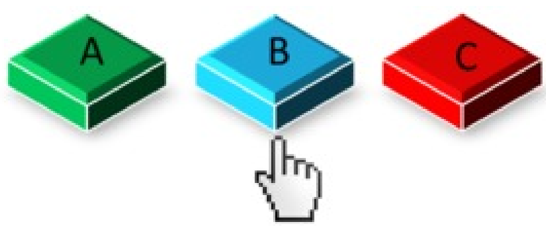
Alignments to Content Standards: 7.SP.C

Task

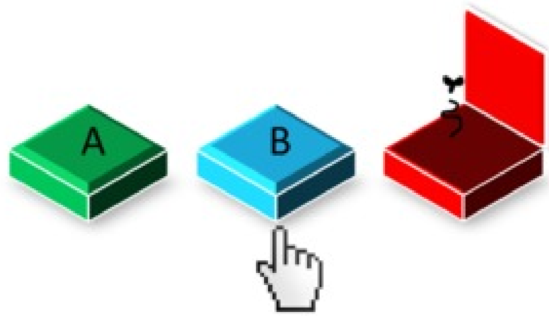
You are playing a video game. At the end of every level, there are three boxes. One contains 10,000 points, and the other two are empty. You are allowed to choose one of the boxes, but before the one you choose opens, one of the other boxes always opens to show that it is empty. The game allows you to either (1) stay with your first choice or (2) switch to the other unopened box.

For example, there are three boxes, A, B, and C.

Suppose you choose box B:

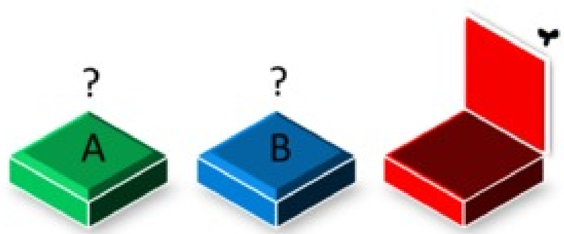


Before box B opens, one of the other boxes opens to show that it is empty. Imagine that in this case it is box C:



The game allows you to stay with your first choice or to switch to the other unopened box.

Should you stay or should you switch?



IM Commentary

The purpose of this task is for students to find the theoretical probability of an event by systematically recording all of the possible outcomes in the sample space and identifying those that correspond to the event. The tricky thing about the problem is that most people assume that it doesn't matter if they stay or switch, thinking that because there are two choices, the chances of getting the points is "50-50." This would be correct if it is equally likely that the 10,000 points is in either of the two boxes, but this is not the case. However, the correct probability can be easily deduced from the uniform probability model applied (appropriately) to the initial situation or approximated by carrying out a simulation. Deciding whether to stay or switch is challenging because there is a temporal element: the player first chooses a box and *then* the computer game reveals what is in a different box. The key insight is that the computer will always open an empty box.

Don't reveal the answer before students have had a chance to struggle with the reasoning here. This struggle is productive and helps students develop insight.

Encourage students to do a simulation of the game (with cups and a ball for example) and after several repetitions they should notice that switching is better than staying. The simulation will likely indicate the correct answer, but students may require more encouragement to provide an explanation: at this point, the teacher might suggest making a list or table of possibilities and, with some careful thinking, this should lead students down the path to understanding. Because the problem is conceptually difficult and involves a game for two people, this task is ideally suited for group work.

Perhaps the easiest way to see the correct probabilities is to observe that if you choose to switch, you will win whenever the 10,000 points are in *either* of the boxes you did not initially choose (2 out of 3). If you stay, you only win the points 1 out of 3 times. Switching exploits the fact that the computer will never open the box with the points in it.

This task illustrates the first four mathematical practices:

MP.1 Make Sense of Problems and Persevere in Solving Them

MP.2 Reason Abstractly and Quantitatively

MP.3 Construct Viable Arguments and Critique the Reasoning of Others

MP.4 Model with Mathematics

Students are presented with a ''real world situation'' in the form of a game and they must provide a probability model accurately analyzing the steps of this game. Because the final result is often counter-intuitive, students will need to actively debate and try to convince one another that these probabilities are correct.

The task is inspired by what has become to be known as the Monty Hall Problem, which is loosely based on the television game show *Let's Make a Deal*, which was popular in the 1960's and 1970's and has been produced intermittently ever since. Contestants were often presented with a choice between three unknown prizes, concealed behind three curtains:

Contestants were asked to choose one of the curtains. The game-show host would then reveal one of the (other) curtains with a low-valued prize. Then the host asks the contestant if they want to switch to the other curtain. Much information about the game can be found here:

http://en.wikipedia.org/wiki/Let's_Make_a_Deal.

A slightly different scenario is presented in a more modern game show, Deal or No Deal: https://en.wikipedia.org/wiki/Deal_or_No_Deal. In this show, the contestant is presented with 26 suitcases, each containing a dollar amount between one cent and one million dollars. The contestant is offered different amounts of money in exchange for her suitcase but can play all the way til the end and keep what is in her suitcase. More importantly, it is the contestant who chooses which of the other 25 suitcases to open, revealing the different dollar amounts inside. In the language of standard 7.SP.7, the difference between these two scenarios is that in Deal or No Deal each choice of a suitcase is a ''chance process'' whereas in Let's Make a Deal it is a deliberate choice made by the game host. Suppose that a contestant in Deal or No Deal makes it down to two suitcases, one containing one million dollars and the other containing one dollar. The contestant has no reason to switch suitcases: each of the remaining two suitcases has a 50 percent chance of containing the one million dollar prize. This makes sense since the contestant has randomly selected two suitcases (in a manner dramatized by the game show), one of which contains the million dollar prize. There is no additional information available and each of the two remaining suitcases is equally likely to contain the one million dollar prize.

This task was designed for an NSF supported summer program for teachers and undergraduate students held at the University of New Mexico from July 29 through August 2, 2013 (http://www.math.unm.edu/mctp/). Simulating the game was a very powerful method for the participants to grapple with this problem.

Solutions

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Solution: 1 Using a table to show equally likely outcomes

First, let’s analyze the case where you first choose box A. There are three possibilities: * There are 10,000 points in box A. Then the computer can open either of the other two boxes because they are both empty. If the computer opens B and you stay, you win. If the computer opens C and you stay, you win. So if you stay with box A, you will win 10,000 points. If you switch, you don’t win any points. * There are 10,000 points in box B. Then the computer can only open box C, because it never opens the box with points in it. If you stay with box A, you don’t win any points. If you switch to box B, you win 10,000 points. * There are 10,000 points in box C. Then the computer can only open box B, because it never opens the box with points in it. If you stay with box A, you don’t win any points. If you switch to box C, you win 10,000 points. The same kind of analysis holds if you choose box B or C at the beginning: if you were right the first time, staying pays off (which happens 1 out of 3 times), and if you are wrong, then switching pays off (which happens 2 out of 3 times). Let’s make a table that summarizes the results of staying.

Staying strategy	10,000 pts in box A	10,000 pts in box B	10,000 pts in box C
You choose A	You win 10,000 pts	You win 0 pts	You win 0 pts
You choose B	You win 0 pts	You win 10,000 pts	You win 0 pts
You choose C	You win 0 pts	You win 0 pts	You win 10,000 pts

Notice that you will win 10,000 points in 3 of 9 equally likely situations, so the probability of winning the points is $\frac{3}{9} = \frac{1}{3}$ if you stay with your initial choice. Now let's make a table that summarizes what happens if you switch.

Switching strategy	10,000 pts in box A	10,000 pts in box B	10,000 pts in box C
You choose A	You win 0 pts	You win 10,000 pts	You win 10,000 pts
You choose B	You win 10,000 pts	You win 0 pts	You win 10,000 pts
You choose C	You win 10,000 pts	You win 10,000 pts	You win 0 pts

Notice that you will win 10,000 points in 6 of 9 equally likely situations, so the probability of winning the points is $\frac{6}{9} = \frac{2}{3}$ if you switch when you have the chance. So if you always switch, you will be more likely to win.

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Solution: 2 abstract reasoning

At the beginning of the game, the 10,000 points is equally likely to be in Box A, B, or C. So there is a $\frac{1}{3}$ chance that the points are in box A, a $\frac{1}{3}$ chance that the points are in box B, and a $\frac{1}{3}$ chance that the points are in box C. This means that there is a $\frac{1}{3}$ chance that the points are in the box you select and a $\frac{2}{3}$ chance that the points are in one of the other two boxes. If you choose to stay, you will win $\frac{1}{3}$ of the time; if you use the switching strategy, however, you will win $\frac{2}{3}$ of the time, making the switching strategy better.

In concrete terms, the reason why the switching strategy is better is that by switching, you win whenever the points are in *either* of the boxes you did not initially choose.